Presented at: The Winter Annual Meeting of The American Society of Mechanical Engineers
COMPUTATIONAL METHODS FOR INFINITE DOMAIN MEDIA-STRUCTURE INTERACTION
Washington D. C., November 15-20, 1981

A NON-REFLECTING BOUNDARY FOR EXPLICIT CALCULATIONS

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ABSTRACT

The dynamic analysis of infinite domains using finite models can be improved through the use of non-reflecting boundaries. These boundaries have been developed and successfully implemented in frequency domain analysis. This paper proposes a method that is applicable to explicit time domain calculations. Two sets of boundary zones with different boundary conditions are used. The boundary conditions are selected such that all reflections are cancelled when the solutions in the two sets of boundary zones are superimposed.

Continuous superposition of the two solutions eliminates reflections as they arise. This avoids the problem of multiple reflections, which otherwise results in an exponential increase of the number of solutions to be superimposed. The governing equations are solved only once in the interior of the model and twice (once for each set of boundary conditions) in a narrow band of elements along the boundary.

The theoretical justification of the method is presented, together with some examples which show the procedure and its implementation to be both accurate and efficient.

INTRODUCTION

The evaluation of the dynamic response of infinite systems using finite numerical models is a problem often encountered in engineering analysis. In such problems, it is usually important to represent properly the wave propagation effects in the infinite medium. If the internal damping of the medium is not high enough to damp out the waves propagating towards any of the artificial boundaries of the finite model, then these waves will be reflected and trapped within the model. The boundary reflection will cause errors in the computed responses. There are two ways to avoid this problem. Firstly, the boundaries of the model can be placed a sufficiently large distance away from the source of excitation such that the radiated waves are absorbed by the internal damping in the medium before they can be reflected off the boundaries. Secondly, special boundary conditions can be imposed such that the boundaries are transparent to the radiated waves.
In problems with low to moderate damping, the first solution may prove to be prohibitively expensive. For example, in the dynamic analysis of a structure resting on a soil layer, Rosset and Ettouney [1] have shown that, to get accurate answers, the lateral boundaries of the soil mesh should be placed at a distance of 5D to 10D away from the edge of the foundation (of diameter D) of the structure. (The diameter of a typical reactor building could be about 40m, and so the required soil mesh must extend at least 200 meters out on each side of the foundation). The second solution of using special boundary conditions permits the use of much smaller meshes and leads to more economical solutions.

Non-reflecting boundaries have been developed and implemented in frequency domain finite element computer models. Lymer and Kuhlmeier [2] and Hitchings and Dance [3] proposed the use of viscous dashpots for the absorption of the radiation waves at the boundaries of a finite mesh. Both papers, through different considerations concluded that viscous dashpots with coefficients related to the impedance of the infinite medium are efficient absorbers of plane elastic waves. Dashpots with constant coefficients are effective absorbers of plane elastic body waves. Here $p$ is the density of the soil medium and $v_y$ and $v_z$ are the shear and compressional wave velocities; $a$ and $b$ are constant coefficients for which both references suggested values of unity.

White et al [4] have shown that dashpots with coefficients $a$ and $b$ as a function of Poisson's ratio give a small improvement over the standard viscous dashpots. For the efficient absorption of surface waves Lymer and Kuhlmeier have shown that the coefficients $a$ and $b$ should be functions of frequency. Consequently, the dashpots with constant coefficients can be used. Rosset and Ettouney [1] indicated that the viscous boundaries represent an improvement over the case of the fixed or free boundary, but it still requires that the boundaries of the mesh be placed a substantial distance away from the source of excitation. For the case of a structure on a soil layer, the lateral edges of the soil mesh should be 5 diameters away from the structure.

An elegant and efficient non-reflecting boundary for frequency domain solutions was proposed by Waas [5]. This so-called consistent boundary solves analytically for the propagation of waves in the unbounded medium and couples this solution to the boundaries of the finite element mesh. The consistent boundary appears to give accurate solutions, but it can be used only in the frequency domain. Another method that can be used in the frequency domain is the infinite element suggested by Belytschko and Zienkiewicz [6]. The method can be used for static as well as for periodic problems [7].

By definition, frequency dependent non-reflecting boundaries cannot be accommodated within the framework of time-domain solutions. Clayton and Engquist [8] suggested a method that is applicable in the time domain and showed that it gives good results for body waves. The method separates the outward and inward moving wave fields through paraxial approximations of the scalar and elastic wave equations. Along the boundary only the outward moving energy is modelled. The effectiveness of the method depends on the angle of incidence of the waves, as in the case of the viscous boundary. The paper does not give any indication of how the method would perform for surface waves.

Smith [9] proposed a method for cancelling out single boundary reflections by superimposing the complete solutions of two independent boundary value problems using Neuman and Dirichlet boundary conditions. The formulation as presented by Smith works both for body and surface waves and is independent of frequency and incidence angles. However it requires $2^N$ complete dynamic solutions if $N$ reflections occur during the time frame of the problem. In addition, the method fails when a given wave is reflected at the same boundary more than once.

The purpose of this paper is to present a practical solution to the boundary reflection problem for explicit time domain analysis. The method proposed uses the superposition principle as adopted by Smith. However, the
boundary conditions are changed from fixed and free to constant velocity and constant stress, and the reflected waves are eliminated as they occur at the boundaries. This latter refinement avoids multiple reflections and the need for F complete solutions. The numerical solution is performed once only, except around the boundary where two solutions are necessary within a narrow band of elements.

This paper presents the theory and numerical implementation of the superposition non-reflecting boundary. A few examples are presented to demonstrate its efficiency.

THEORETICAL FORMULATION

As mentioned earlier, the theoretical basis of the superposition boundary was first presented by Smith [9]. The method consists of superimposing the solutions to two boundary value problems which differ only on the boundary conditions. Smith stated that, by suitably selecting the two sets of boundary conditions, all elastic reflections could be exactly cancelled. He further showed mathematically how this happened for the case of the scalar wave equation and for that of plane compressional waves arriving at the boundary of a two-dimensional continuum. A three-dimensional formulation will be presented in this section.

Governing Equations

Consider a three-dimensional isotropic linearly viscoelastic continuum with constant moduli. This condition does not need to be fulfilled everywhere in the model but only locally in the neighbourhood of the non-reflecting boundary. The constitutive relations are:

\[ \sigma_{ij} = \lambda \delta_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \]  

(1)

where \( \sigma_{ij} \) is the stress tensor, \( \epsilon_{ij} \) is the strain tensor, \( \lambda \delta_{kk} \) are the two complex Lamé constants, \( \delta_{ij} \) is the Kronecker delta.

The field equations, representing the conservation of linear and angular momentum, are in the absence of body torques:

\[ \sigma_{ij, j} + \rho f_{ij} = 0 \]

\[ \epsilon_{ij} = \epsilon_{ij} \]

(2)

where \( \rho \) is the mass density, \( f_{ij} \) are the body forces per unit mass, \( \epsilon_{ij} \) represents the displacement vector.

The strain tensor is related to the displacements by the well known relationships:

\[ \epsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \]

(3)

It will be convenient to make use of Helmholtz' theorem to express the displacement field in terms of potentials:

\[ u_{i} = \delta_{i} + \epsilon_{ijk} \phi_{j,k} \]

(4)

Tensor notation is used throughout this section. Only i, j, k are running indices, other symbols refer to specific values of the index. Repeated indices imply summation over the range of the index, 1 to 3 unless otherwise stated. Commas represent spatial derivatives and dots imply time derivatives.

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where $\phi$ is the scalar potential

$\mathbf{v}_\mathbf{i}$ is the solenoidal vector potential

$\varepsilon_{ijk}$ is the permutation tensor

Boundary Conditions

Let $x_i$ be an orthogonal system of coordinates and, without loss of
generality, take $x_3 = 0$ to be the planar boundary intended to be non-reflecting;
b is clearly one of the values of the index $i$ from its range of 1 to 3.
The two sets of boundary conditions to consider are designated I and II,
respectively, and can be described as follows:

1.a: Fixed in the normal direction: $u_b = 0$
1.b: Free in all tangential directions $\tau_{ib} = 0, \mathbf{v}_i \mathbf{v}_b$. Since $\delta_{ib} = 0,$
$\mathbf{v}_i \mathbf{v}_b$, this second condition implies $\tau_{ib} = 0$ via eq 1. And, since
$\eta_i = 0$, this implies $\eta_{ib} = 0$ in order to satisfy eq 3. These
boundary conditions are therefore expressed in terms of displacements as:

$u_b = 0$ ...

$u_{i,b} = 0, \mathbf{v}_i \mathbf{v}_b$ ...

This set of conditions can be readily expressed in terms of potentials:

$\phi_{,b} + \varepsilon_{bkl} k_{,kl} = 0$ ...

$\phi_{,ib} + \varepsilon_{ijkl} k_{,jl} = 0, \mathbf{v}_i \mathbf{v}_b$ ...

II.a: Fixed in all tangential directions: $\mathbf{u}_i = 0, \mathbf{v}_i \mathbf{v}_b$.
II.b: Free in the normal direction $\tau_{bb} = 0$. Notice that condition II.a
implies $\eta_{i,k} = u_{b,b}$, hence $\tau_{kk} = u_{b,b}$. Since $\epsilon_{i,b} = u_{b,b}$ as well,
condition II.b can be expressed using eq 1 as $u_{b,b} = 0$.
The second set of boundary conditions can therefore be written in
terms of displacements as:

$u_i = 0, \mathbf{v}_i \mathbf{v}_b$ ...

$u_{b,b} = 0$ ...

and in terms of potentials:

$\phi_{,i} + \varepsilon_{ijkl} k_{,jl} = 0, \mathbf{v}_i \mathbf{v}_b$ ...

$\phi_{,bb} = 0$ ...

The two sets of boundary conditions, I and II, are now represented by eqs
7, 8 and 11, 12 respectively. An attempt will be made to show that superposition
of the two solutions to eqs 1 to 3 with boundary conditions I and II result in
cancellation of all reflections.

P-wave Incidence

The case of an incident compressional wave will be considered first. The
potentials associated with a unit incident compressional plane wave and its
hypothetical compression and shear wave reflections can be written as follows:

$\phi = \exp \left\{ i \frac{k}{v_p} x_1 - t \right\} + A \exp \left\{ i \frac{k}{v_p} x_1 - t \right\}$ ...

$\mathbf{v}_i = B_i \exp \left\{ i \frac{\gamma_k}{v_s} x_k - t \right\}$ ...

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where \( i \) is \( \sqrt{-1} \) when not used as an index
\( \alpha, \beta, \gamma \) are the directional cosines of the propagation path
\( \omega \) is the angular frequency
\( v_p, v_s \) are the velocities of propagation of P- and S-waves, respectively
\( t \) is time

Notice that not all three components of \( B \) are completely independent. In fact, it can be assumed that \( B_y = 0 \) without loss of generality. For if \( B^*_i = B_i + r \gamma_i \) where \( r \) is arbitrary, \( B_i^* \) and \( B_i \) would produce identical displacement fields.

Substitution of \( \xi \) and \( \psi \) from eqs. 13, 14 into the boundary condition 1.a generates the following equation:

\[
\omega \frac{\alpha_b}{v_p} \exp \left\{ i \omega \left( \frac{\xi}{v_p} x_1 - t \right) \right\} + A \omega \frac{\beta_b}{v_p} \exp \left\{ i \omega \left( \frac{\xi}{v_p} x_1 - t \right) \right\} +
\]
\[
+ \epsilon_{ijk} \frac{\alpha_b}{v_p} \frac{\gamma_j}{v_s} \exp \left\{ i \omega \left( \frac{\xi}{v_s} x_1 - t \right) \right\} = 0 \quad \ldots(15)
\]

Since eq 15 must be fulfilled for any combination of values of \( x_1 \) such that \( x_b = 0 \):

\[
\frac{\alpha_b}{v_p} = \frac{\beta_b}{v_p} = \frac{\gamma_j}{v_s}, \quad \text{Vi} \# b \quad \ldots(16)
\]

making the exponents identical; this also implies that, since the directional cosines are components of a unit vector:

\[
|\alpha_b| = |\beta_b|
\]
\[
|\gamma_j| = \left[ 1 - \left( \frac{\xi}{v_p} \right)^2 \left( 1 - \frac{\alpha_b^2}{v_p} \right) \right]^{1/2} \quad \ldots(17)
\]

It is clear that \( \beta_b = -\alpha_b \) since \( \beta_b - \alpha_b \) corresponds to the incident and not the reflected wave. Notice that eqs 16 and 17 are a three-dimensional statement of Snell's law. For any given set of \( (x_1, t) \), eq 15 reduces to:

\[
\frac{\alpha_b}{v_p} + A \frac{\beta_b}{v_p} + \epsilon_{ijk} \frac{\gamma_j}{v_s} = 0 \quad \ldots(18)
\]

When \( \xi \) and \( \psi \) are substituted into the boundary condition 1.b, eqs 16 and 17 follow in a similar fashion. The condition left is then:

\[
\frac{\alpha_b}{v_p} + A \frac{\beta_b}{v_p} + \frac{\gamma_b}{v_s} + \epsilon_{ijk} \frac{\alpha_b}{v_s} = 0, \quad \text{Vi} \# b \quad \ldots(19)
\]

Using the previous Snell relationships, eqs 18, 19 can be written as:

\[
\frac{v_s}{v_p} \frac{\alpha_b}{v_p} (1-A) + \epsilon_{ijk} \frac{\beta_b}{v_s} = 0 \quad \ldots(20)
\]
\[
\left( \frac{v_s}{v_p} \right)^2 \frac{\alpha_b}{v_p} (1-A) + \epsilon_{ijk} \frac{\gamma_b}{v_s} = 0, \quad \text{Vi} \# b \quad \ldots(21)
\]

which, together with the relationship \( B_i \gamma_i = 0 \) constitute a set of four
independent linear homogeneous equations with four unknowns $A$, $B$. All four
unknowns must therefore be equal to zero. The consequences are $A = 1$ (a
reflected P-wave in phase with the incident one) and $B = 0$ (no reflected S-
waves).

The second set of boundary conditions must now be considered. Substitution
of $\psi$ and $\psi_1$ into condition II.a yields once again eqs 16, 17 and:

$$
\frac{a_i}{v_p} + A \frac{g_i}{v_p} + e_{ijk} B_k \frac{v_j}{v_s} = 0 , \quad \text{Wishb} \tag{22}
$$

Similarly, when introducing $\psi$ and $\psi_1$ into condition II.b, eqs 16, 17 are
implied and, further:

$$
\frac{a^2}{v_p} + A \frac{b^2}{v_p} = 0 \tag{23}
$$

As before, eqs 22, 23 can be written using eqs 16, 17 as:

$$
\frac{v_s}{v_p} a_i (1 + A) + e_{ijk} B_k \frac{v_j}{v_s} = 0 , \quad \text{Wishb} \tag{24}
$$

$$
\frac{a^2 (1 + A)}{v_p} = 0 \tag{25}
$$

which, together with $B_i Y_i = 0$, make the set of four independent linear
homogeneous equations which imply that $1 + A$ and $B_i$ vanish simultaneously.
This corresponds to $A = -1$ (a reflected P-wave exactly out of phase with the incident
one) and $B = 0$ (no reflected S-waves).

In summary, the superposition of the solutions from the two sets of
boundary conditions results in two P-waves which cancel each other exactly and
no shear waves.

S-wave Incidence

Consider an incident shear wave and its potential shear and compressional
wave reflections. The associated potentials are:

$$
\psi = C \exp \left\{ \frac{i \omega}{v_p} \frac{Y_i}{v_s} x_i - t \right\} \tag{26}
$$

$$
\psi_i = A_1 \exp \left\{ \frac{i \omega}{v_s} \frac{a_k}{v_s} x_k - t \right\} + B_1 \exp \left\{ \frac{i \omega}{v_s} \frac{b_k}{v_s} x_k - t \right\} \tag{27}
$$

It should be remembered that $A_1 = C$ and $B_1 = 0$ without loss of
generality. Substitution of these potentials into the boundary condition I.a
yields the following equations:

$$
i \omega \frac{Y_i}{v_p} C \exp \left\{ \frac{i \omega}{v_p} \frac{Y_i}{v_s} x_i - t \right\} + e_{ijk} A_k \frac{i \omega}{v_s} \frac{a_j}{v_s} \exp \left\{ \frac{i \omega}{v_s} \frac{a_j}{v_s} x_j - t \right\} +

+ B_k \frac{i \omega}{v_s} \frac{b_k}{v_s} \exp \left\{ \frac{i \omega}{v_s} \frac{b_k}{v_s} x_k - t \right\} = 0 \tag{28}
$$

Again this results in Snell's law:
\[ \frac{\gamma_i}{v_p} = \frac{\alpha_i}{v_s} = \frac{\beta_i}{v_s}, \quad \forall i \neq b \quad \ldots(29) \]

as well as the relationship:

\[ C \frac{\gamma_b}{v_p} + e_{b,jk} \left[ A_k \frac{\alpha_k}{v_s} + B_k \frac{\beta_k}{v_s} \right] = 0 \quad \ldots(30) \]

Snell’s law is recovered again from substitution on the boundary condition 1.b. Besides,

\[ C \frac{\gamma_b}{v_p} + e_{b,jk} \left[ A_k \frac{\alpha_k}{v_s} + B_k \frac{\beta_k}{v_s} \right] = 0, \quad \forall i \neq b \quad \ldots(31) \]

Using Snell’s law, eqs 30, 31 can be rewritten as:

\[ C \frac{v}{v_p} \gamma_b = e_{b,jk} (A_k \frac{\alpha_k}{v_s} + B_k \frac{\beta_k}{v_s}) = 0, \quad \forall i \neq b \quad \ldots(32) \]

\[ C \frac{v}{v_p} \gamma_b + e_{b,jk} (A_k \frac{\alpha_k}{v_s} + B_k \frac{\beta_k}{v_s}) = 0, \quad \forall i \neq b \quad \ldots(33) \]

Two more equations are available, namely \( A_i = 0 \) and \( B_i = 0 \); in fact, one of the two is redundant. Remembering that \( \delta_i = \delta_s \), \( \forall i \neq b \), it is not difficult to notice that the above system of four independent equations is linear and homogeneous on the four unknowns \( C, A_b, B_b \) and \( A_iB_i, \forall i \neq b \). All four unknowns must therefore vanish.

As a consequence, \( C = 0 \) (no P-wave reflection); also, \( B_i = A_i \), \( \forall i \neq b \) and \( B_b = A_b \) represent the amplitude of the S-wave reflected in relation to the incident wave amplitude.

Boundary condition II.a yields again Snell’s law and the relation:

\[ C \frac{\gamma_i}{v_p} = e_{i,jk} \left[ A_k \frac{\alpha_k}{v_s} + B_k \frac{\beta_k}{v_s} \right] = 0, \quad \forall i \neq b \quad \ldots(34) \]

while boundary condition II.b immediately gives:

\[ C \frac{v}{v_p} = 0 \quad \ldots(35) \]

As in the previous case, eqs 34, 35 together with the complementary equation \( A_i = 0 \) become, using Snell’s law, a system of four independent homogeneous linear equations. The four unknowns \( C, A_b, B_b \), and \( A_iB_i, \forall i \neq b \) are therefore zero. The result is \( C = 0 \) (no P-wave reflection); moreover, \( B_i = A_i \), \( \forall i \neq b \) and \( B_b = A_b \) (the S-wave reflected is exactly opposite to the one associated with boundary conditions 1).

Therefore, superposition of the solutions to the two boundary value problems results in two shear waves which cancel each other exactly and no compressional waves.

**Surface Wave Incidence**

If the model has a free surface, it is in principle possible that Rayleigh waves will be generated, propagated and will eventually reach the non-reflecting boundary, if the latter intersects the free surface. Further, if the medium is
not homogeneous, but there is some increase in stiffness with distance to the free surface, Love waves are also likely to arise.

For the purpose at hand, Love waves are shear waves with the particle motion parallel to the free surface. As such, the conclusions previously derived for shear waves can still be applied. The superposition boundary hence absorbs incident Love waves without reflections.

Since Rayleigh waves cannot propagate along the superposition boundary, an incident Rayleigh wave will only result in one Rayleigh wave reflection and some body wave reflections (Richart et al. [10]). The potentials associated with a Rayleigh wave propagation along a free surface defined by $k_s = 0$ are:

$$\phi = A \exp \left\{ -p x_s + i\omega \left( \frac{\alpha}{v} x_1 - t \right) \right\} \quad \ldots(36)$$

$$\psi = B \exp \left\{ -q x_s + i\omega \left( \frac{\alpha}{v} x_k - t \right) \right\} \quad \ldots(37)$$

where $\alpha_s = E_s = 0$ and, as a consequence of $\tau_{is} = 0$ at $x_s = 0$:

$$p = \frac{\omega}{v} \left[ 1 - \left( \frac{\alpha}{v} \right)^2 \right] \frac{1}{p} \quad \ldots(38)$$

$$q = \frac{\omega}{v} \left[ 1 - \left( \frac{\alpha}{v} \right)^2 \right] \frac{1}{q} \quad \ldots(39)$$

$$\frac{B}{A} = -\frac{2i \omega}{q} \frac{p^2}{\omega} \quad \ldots(40)$$

and where $v$ is the real root of the Rayleigh equation comprised between zero and $v_p$:

$$\left[ 1 - \left( \frac{\alpha}{v} \right)^2 \right]^2 = 4 \left[ 1 - \left( \frac{\alpha}{v} \right)^2 \right] \left[ 1 - \left( \frac{\alpha}{v} \right)^2 \right]$$

The potentials associated with the possible reflections at the non-reflecting boundary are composed of incident and reflected Rayleigh waves as well as reflected body waves. Apart from the alternative I and II boundary conditions, the free surface must remain stress free:

$$\lambda u_{ik} + \nu (u_{ik, k} + u_{ik, i}) = 0 \quad \ldots(42)$$

The application of I and II boundary conditions is performed as in the previous case of body wave incidence. It is not difficult to see that the two corresponding Rayleigh wave reflections fulfill Snell’s law and cancel each other exactly.

It also appears that no $P$- or $S$-wave reflections survive the superposition of solutions, as it is verified by examples described in further sections. But, although it is not difficult to show that $P$- and $S$-wave paths are related by Snell’s law, the authors have not yet achieved a formal proof of their inexistence or cancellation upon superposition.

**Boundary Waves Along The Superposition Boundary**

As stated by Smith [8], Rayleigh waves cannot propagate along a superposition boundary. The reason is that the projections of the boundary particle displacements on both the boundary and its normal would have to be different from zero. But each set of boundary conditions requires one of these two projections to be zero. Hence, Rayleigh waves cannot exist at the superposition boundary.
Love-type waves will appear and propagate along this boundary only if there is an increase in stiffness with distance from the boundary, an unlikely condition in normal problems. Even in that situation, their exponential decrease in amplitude with distance from the boundary will result in insignificant effects inside the model [9].

**Velocity Boundary Conditions**

The two sets of boundary conditions (eqs 5.6 and eqs 9.10) have been stated in terms of displacements. However, it is interesting to notice that the cancellation of reflections is identically achieved if velocity boundary conditions are specified. The conditions would be:

1.a: \( \dot{u}_b = 0 \)

1.b: \( \ddot{u}_b \neq 0 \)

11.a: \( u^1_b = 0 \)

11.b: \( \dot{u^1}_b = 0 \)

The expressions in terms of potentials would be identical to eqs 7.8, 11 and 12 except for the dots (time derivations) that would crown every \( \dot{u} \) and \( \dot{u^1} \). All subsequent derivations still hold. The only difference would be the appearance of the (-iω) factor in all expressions as a result of the time differentiation; this factor is, however, of no concern since the right hand side of all the relevant equations is zero. Therefore, the same conclusions are derived with velocity boundary conditions as were obtained for the corresponding displacement conditions.

In reality, there is no great difference between the two types of conditions. Strictly speaking, they may only differ on the initial (\( t=0 \)) values of \( u \) and \( u^1 \), that is, the initial displacements and displacement gradients at the boundary. The formulation of the boundary value problem is actually identical to the previous one, except that the boundary conditions are no longer homogeneous. In practice, the velocity formulation is of interest to numerical schemes which update nodal coordinates as the calculation progresses, thus reproducing large strain behaviour via a series of small linear steps. In such schemes, even if the boundary displacements and gradients are sufficiently small to remain within the realm of linear elasticity, the continuous references to zero (rather than to zero time derivatives) result in a continuous generation of small shocks. These shocks are very small but their generation is particularly unwanted because they are not exactly equal (and hence do not cancel exactly) for the I and II boundary conditions.

The implementation of the superposition boundary described in further sections is formulated in velocities rather than displacements.

**BOUNDARY DESCRIPTION**

As seen from the previous section, the superposition of the solutions corresponding to the fixed and free boundary conditions cancels the reflected waves. However, as Smith [5] pointed out, if a reflection occurs \( n \) times within the time frame of a problem then it is necessary to superimpose the solutions from \( n \) complete analyses to obtain the correct solution. To overcome this serious handicap, it is proposed to cancel the reflections at the boundaries as they occur, thereby limiting \( n \) to unity.

In a time domain explicit calculation procedure, the computation time-step is short enough that information travels less than one zone of the discretised mesh in one time-step. Consequently, it is possible that by superimposing solutions every time-step, the reflections will be eliminated before the reflected wave can leave the zone next to the boundary. Thus the reflection is cancelled at source. However, in practice, it appears that improved results are obtained if the reflections are cancelled in a region of three to four boundary zones rather than one. Further investigation into this aspect of the method is
under way, and will be reported at a later date.

To implement the superposition boundary method, two overlapping sets of boundary zones A and B (each three or four zones wide) are connected independently to the main calculation grid as depicted in Figure 1. A wave that propagates from the main mesh enters the two boundary zones simultaneously. This wave will reflect off the boundaries of the regions A and B. By using the appropriate boundary conditions at these boundaries, the reflections are eliminated by averaging the stresses and velocities in the two boundary regions. This superposition of the two boundary regions is performed sufficiently regular such that the reflections cannot propagate out of the boundary regions.

![Figure 1: Boundary zones and main mesh for the superposition boundary](image_url)
The grid prints M1, M2 ... of the main mesh (referred to as master nodes) are connected directly to the boundary nodes A1, A2 ... and B1, B2 ... etc. (slave nodes). To transmit the motion from the main mesh to the boundary zones, the slave nodes are assigned the same velocities as the corresponding master nodes:

\[ v_{A,B} = v_{M} \]  \hspace{1cm} ...(47)

For a seismic analysis this relationship is modified to reflect the free-field seismic motion as discussed later.

The waves reflected off the boundaries of regions A and B must not be allowed to propagate into the main mesh. Consequently, the solutions should be superimposed at time intervals of at least \( t_m \), where \( t_m \) is the time taken for the waves to traverse the regions A or B. For a regular grid of equal element sizes, the superposition should be performed every \( n \) time-steps of calculation when \( n \) is the number of columns of zones in the boundary region.

The superposition procedure averages the forces, stresses and velocities of the two boundary zones. Consider, the problem at the start of the analysis, when the stresses and velocities in the boundary zones are zero. The conditions at the two boundaries are as follows:

1. **REGION A**: fixed in x-direction
   free in y-direction
2. **REGION B**: fixed in y-direction
   free in x-direction

This is illustrated in Figure 2.

**Figure 2**: Initial conditions for boundary regions

To superimpose the solutions of these two zones the 'roller' restraints have to be replaced by equivalent reaction forces. In the A region the external equivalent force at each node is that force required to hold the x-velocity of that node to zero at the instant of adding. The force at each node is computed as equal and opposite to the resultant force produced by the internal stresses on that node. Similarly, in the B region the reaction forces in the y-direction are to be computed. The superposition principle can then be applied shown in Figure 3.
The stresses, velocities and reaction forces are averaged to give the 'true' solution as follows:

Boundary forces:
- \( F_{Ax} / 2 \) - \( x \) direction
- \( F_{Ay} / 2 \) - \( y \) direction

Zone stresses:
- \( (c_{Ax} + c_{Bx}) / 2 \) - direct \( x \) stress
- \( (c_{Ay} + c_{By}) / 2 \) - direct \( y \) stress
- \( (c_{Axy} + c_{Bxy}) / 2 \) - shear stress

Nodal point velocities:
- \( (V_A + V_B) / 2 \) - all velocities

It is clear that after the first superposition cycle, the boundaries of regions A and B can no longer be considered as fixed-free. By virtue of the procedure, the boundary nodes now have non-zero velocities and non-zero forces. For the next and all subsequent cycles of superposition, therefore, the boundary conditions are adjusted from fixed-free to be:

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REGION A: constant velocity in x-direction  
constant force in y-direction  
REGION B: constant velocity in y-direction  
constant force in x-direction  

At the time of applying the superposition procedure the second time, the boundary forces required to hold the x-velocity in Region A and the y-velocity in Region B constant are computed and the same procedure as described previously is applied. The method then repeats itself for all subsequent superposition cycles.

COMPUTER IMPLEMENTATION

The procedure described above can be easily implemented in explicit finite element or finite difference codes. Once the mesh idealisation has been set up together with the two boundary regions, the computer algorithm for analysis can be written as follows:

(i) Determine the number of cycles of calculation between superposition - say m  
(ii) Assign the appropriate conditions to the boundaries of regions A and B  
(iii) Perform the analysis calculations in the main mesh and boundary regions A and B  
(iv) Every mth time-step of calculation superpose the results in regions A and B, by first computing the boundary forces and then averaging the velocities, stresses and boundary forces in the two boundaries. Assign the new boundary conditions to regions A and B  
(v) Increment the time and repeat from step (iii)

This procedure has been incorporated into the computer programme PRESS (Principia's Explicit Soil−Structure interaction program) in which the soil calculation is carried out using an explicit finite difference method as in programmes like PISCES, STEALTH etc. In this programme the non-reflecting boundary formulation is used to simulate the semi-infinite soil continuum. A summary flow chart of the procedure in PRESS is given in Figure 4.

APPLICATION TO SEISMIC SOIL-STRUCTURE INTERACTION

In seismic soil-structure interaction problems, it is common practice to apply the input excitation along the base of the soil mesh and to assume a vertical propagation of waves through the soil. If the superposition non-reflecting boundary is implemented at the lateral edges of a soil mesh, then the formulation must be such that it reacts only to the waves radiating away from the structure and not to motion of the master nodes resulting from the vertical propagation of the seismic input.

In the absence of a structure, the assumption of vertical propagation of waves implies a one-dimensional soil response and the lateral non-reflecting boundaries should not participate in the computations. When a structure is connected to the soil, any deviation from the one-dimensional response will be the result of soil-structure interaction and the radiation of waves away from the structure. It is logical, therefore, to permit the non-reflecting boundaries to react only to any response different to the one-dimensional response of the soil mesh. To achieve this within the framework of the present formulation the relationship between the master and slave nodes as expressed in equation (47) is modified to be

$$V_{A,B} = V_M - V_{1D}$$  

...(48)

when $V_{1D}$ is the vector of the 1-dimensional response of the soil mesh. The 1-D analysis can be carried out in parallel with the soil-structure interaction analysis as proposed in the paper by Kumar and Rodríguez-Ovejero [1].

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Figure 4: Summary flowchart for the computer implementation of the superposition boundary

EXAMPLES

Three examples are presented here to demonstrate the accuracy of the superposition non-reflecting boundary formulation. The first example considers a one-dimensional wave propagation along a bar. In the second example a point load is applied at the surface of a single soil layer on rigid bedrock. The steady state displacement response along the surface of the soil is computed and compared to results obtained using the consistent boundary. The third example considers a two-dimensional soil-structure interaction analysis with seismic loading. All analyses were performed using the computer programme PRESS.

Example 1: One-dimensional wave propagation

In this example, the two-dimensional computer programme was used to model a one dimension wave propagation along an infinite bar. A model of the bar is shown in Figure 5. One row of 20 elements was used to model a 200m length of bar. For a one-dimensional response, the y-degrees-of-freedom at all nodes were fixed. The material was assumed to have a compressional wave velocity of 1000 m/sec and a Poisson's ratio of 0.3. The density was taken as 2000 kg/m³. Also zero damping was assumed. A velocity pulse of maximum amplitude 0.1 m/sec was specified at and A of the bar, as shown in Figure 5. The x-velocity responses at nodes 90 metres (point B) and 190 metres (point C) away from the source of excitation (point A) were computed and plotted in Figure 5. The analysis was performed for a sufficiently long time (1 second) to observe any reflections from the boundary. From Figure 5, it is obvious that the superposition boundary simulates almost perfectly the infinite bar. The small dynamic oscillations at the end of the main pulse responses can be attributed to the coarseness of the mesh discretisation. It is expected that these will become smaller and smaller as the element size is progressively decreased.

Example 2: Point load on a layer

This second example considers a harmonic point loading on the surface of a layer of finite depth but of infinite lateral extent. The layer is in deep, and has the following material properties:
Figure 5: One dimensional wave propagation
Shear Modulus: \( G = 1.0 \text{ N/m}^2 \)
Density: \( \rho = 1.0 \text{ kg/m}^3 \)
Poisson's ratio: \( \nu = 0.33 \)
Damping: \( \beta = 0.05 \)

The base of the model is rigid and a unit vertical harmonic force is applied at the surface of layer as shown in Figure 6. In the analysis the non-reflecting boundary was placed 1.4m away from the point of loading. The steady-state displacement response at distances 0.1, 0.2, 0.3, 0.4 and 0.5m from the loadpoint on the surface of the layer were computed. A total of 3 analyses were performed:

1. Loading frequency 0.5Hz - superposition boundary
2. Loading frequency 0.5Hz - free boundary
3. Loading frequency 1.0Hz - superposition boundary

The first two analyses compare the amplitudes and phases of the computed steady-state responses for the superposition and for the free boundaries. In the superposition analysis, the boundary was placed at a distance of 1.4m with the boundary regions 0.4m wide. For a direct comparison, the free boundary was placed at a distance of 1.8m. Figure 7 compares the free boundary solution with the superposition boundary solution for the excitation frequency of 0.5Hz. The corresponding response using the consistent boundary Roesset [12] is also plotted in Figure 7. The results indicate an excellent agreement between the consistent and superposition boundaries. The free boundary solution is very different and demonstrates quite forcefully the accuracy of the non-reflecting boundary.

The steady-state displacement responses for the superposition and consistent boundaries for an excitation frequency of 1.0Hz are compared in Figure 8. Again the agreement is very good.

**Example 3: Soil-structure interaction example**

This soil-structure interaction example was presented elsewhere [1]. However, it is repeated here for the sake of completeness. In-soil-structure interaction analyses it is convenient and efficient to use the superposition non-reflecting boundaries at the lateral edges of a model and viscous dashpots at the base, as explained in reference [1]. This type of model was used in this example.

Figure 9 presents a diagram of the soil-structure model used in this analysis. Three models were considered:

1. Model A - small mesh with lateral superposition boundaries and with viscous dashpots at the base.
2. Model B - large mesh (320m x 100m) without non-reflecting boundaries.
3. Model C - small mesh without non-reflecting boundaries.

In models B and C, the lateral boundaries were fixed and the seismic excitation was specified as acceleration time-histories at the base. Model A used stress input histories at the base (see Ref. [13]).

The soil was assumed to be isotropic, homogenous and elastic with the following properties:

Density: \( \rho = 2000 \text{ kg/m}^3 \)
Shear Modulus: \( G = 3.2 \times 10^7 \text{ N/m}^2 \)
Poisson's Ratio: \( \nu = 0.2 \)
Damping: \( \beta = 5\% \)

The structural model of the reactor building has 16 nodes (68 degrees-of-freedom) with nodes 1 - 9 rigidly connected to the soil. Simultaneous horizontal and vertical seismic excitations were prescribed at surface level as being the first 7 secs of artificially generated 30-s earthquakes.
Figure 6: Model for example 2.

MATERIAL PROPERTIES

\[ V = 1.0 \]
\[ V = 0.33 \]
\[ \rho = 1.0 \]
Figure 7: Results for example 2 -
Excitation frequency = 0.5 Hz
Figure 8: Results for example 2 -
Excitation frequency = 1.0 Hz
Figure 9: Models for example 3.
Figure 10: Results from example 3.
Table I compares the peak horizontal and vertical acceleration responses at selected nodes in the structure (1, 4, 10, 16) for the three different soil models. The response spectra for nodes 4 and 16 are presented in Figure 10. The results show remarkable similarity between models A and B. These results are consistent with the findings of the authors in a number of other examples, and they demonstrate quite clearly the accuracy of the non-reflecting boundaries. It should be pointed out that the accuracy is just as good in examples in which the soil profile is horizontally layered with different material properties in each layer.

Table 1. Peak acceleration responses obtained from soil-structure interaction analyses

<table>
<thead>
<tr>
<th>Node Number</th>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
</tr>
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<tr>
<td>Horizontal</td>
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<td>0.31</td>
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<tr>
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<td></td>
<td></td>
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<tr>
<td>16</td>
<td>0.21</td>
<td>0.21</td>
<td>0.37</td>
</tr>
</tbody>
</table>

Model A - small mesh with non-reflecting boundaries
Model B - large mesh without non-reflecting boundaries
Model C - small mesh without non-reflecting boundaries

REFERENCES


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